

LET $\Pi(t) = V(t) - S(t) \cdot \Delta(t)$ BE THE HEDGE PORTFOLIO.

THEN

$$d\Pi(t) = dV(t) - d(S(t) \cdot \Delta(t))$$

USE ITO PRODUCT RULE
(COROLLARY 4.6.3)

$$= \left(\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} ds + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \right) - \left(S(t) \cdot d\Delta + ds \cdot \Delta(t) + ds \cdot d\Delta \right)$$

$\Delta = \frac{\partial V}{\partial S}$

$$\therefore d\Pi(t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt - (S(t) \cdot d\Delta + ds \cdot d\Delta) \quad (*)$$

LET $X(t)$ BE THE REPLICATING STRATEGY FOR $V(t)$,
WHERE $X(t)$ HOLDS $\Delta(t) = \frac{\partial V}{\partial S}$ SHARES OF STOCK,
WITH BORROWING OR LENDING IN THE RISKLESS ASSET,
AS NEEDED, TO SELF-FINANCE THE POSITION.

THE INVESTMENT IN THE RISKLESS ASSET (SKREVE CALLS IT A MONEY MARKET A/C) IS THUS

$$X(t) - \Delta(t) \cdot S(t) = V(t) - \Delta(t) \cdot S(t) = \Pi(t).$$

LET $M(t) = e^{rt}$ BE THE PRICE OF ONE SHARE OF THE RISKLESS ASSET. THEN THE NUMBER OF SHARES IN THE RISKLESS ASSET USED IN THE REPLICATING STRATEGY IS

$$\Gamma(t) = \frac{\Pi(t)}{M(t)} = e^{-rt} \Pi(t)$$

THE CONTINUOUS-TIME SELF-FINANCING CONDITION (4.10.15) IS

$$S(t) \cdot d\Delta + ds \cdot \Delta + M(t) \cdot d\Gamma + dM \cdot \Gamma = 0. \quad (**)$$

WE WANT TO PLUG IN FOR M , dM , & $d\Gamma$.

$$M(t) = e^{rt} \Rightarrow dM = r e^{rt} dt = r M(t) dt \quad (2)$$

$$\Gamma(t) = e^{-rt} \Pi(t)$$

$$\begin{aligned} \Rightarrow d\Gamma &= -r e^{-rt} \Pi(t) dt + e^{-rt} d\Pi - r e^{-rt} dt d\Pi \\ &= -r \frac{\Pi(t)}{M(t)} dt + \frac{d\Pi}{M(t)} - r \frac{dt \cdot d\Pi}{M(t)} \end{aligned}$$

SO THE CTSF CONDITION (***) FIELDS

$$\begin{aligned} S(t)d\Delta + ds \cdot d\Delta + M(t) \left[-r \frac{\Pi(t)}{M(t)} dt + \frac{d\Pi}{M(t)} - r \frac{dt \cdot d\Pi}{M(t)} \right] \\ + r M(t) \cancel{dt} \left(-r \frac{\Pi(t)}{M(t)} \cancel{dt} + \frac{d\Pi}{M(t)} - r \frac{dt \cdot d\Pi}{M(t)} \right) = 0 \end{aligned}$$

$$\therefore S(t)d\Delta + ds \cdot d\Delta - \boxed{r \Pi(t) dt} + d\Pi - r dt \cdot d\Pi + r dt d\Pi = 0$$

SO, USING (*), WE GET

$$\begin{aligned} \cancel{S(t) \cdot d\Delta + ds \cdot d\Delta} - \boxed{r \Pi(t) dt} + \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \\ - \cancel{(S(t) \cdot d\Delta + ds \cdot d\Delta)} = 0 \end{aligned}$$

$$\text{AND THUS } r \Pi(t) dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \quad (4.10.18).$$

$$\text{NOW PLUG IN } \Pi(t) = V(t) - S(t) \cdot \Delta(t) = V(t) - S(t) \frac{\partial V}{\partial S}$$

AND CANCEL OUT dt FROM EACH SIDE TO GET

$$r \left(V - \frac{\partial V}{\partial S} S \right) = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2},$$

AND REARRANGE TO GET

$$\frac{\partial V}{\partial t} + r S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r V = 0 \quad \left(\begin{array}{l} \text{B.SCHOLES} \\ \text{PDE} \end{array} \right)$$